On new Versions of the Lindeberg-Feller's Limit Theorem

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Abstract

It is well known that the classical Lindeberg condition is sufficient for validity of the central limit theorem. It will be also a necessary if the summands satisfy the condition of infinite smallness (Feller's theorem). The limit theorems for the distributions of the sums of independent random variables which do not use the condition of infinite smallness were called non-classical.

In this paper a non-classical version of Lindeberg-Feller's theorem is given. The exact bounds for the Lindeberg, Rotar characteristics using the difference of the distribution of sum of independent random variables and a standard normal distribution are established. These results improve Feller's theorem.

Keywords: the central limit theorem, the conditions for uniform infinite smallness, the nonclassical theorem of Lindeberg-Feller, characteristics of Lindeberg, Rotar, Ibragimov-Osipov-Esseen

Introduction

Let $X_{n1}, X_{n2}, ..., X_{nn}$, $n = 1, 2, ...$ - be an array of independent random variables (r.v.'s).

Assume that

$$EX_{nj} = 0, \quad EX_{nj}^2 = \sigma_{nj}^2, \quad j = 1, 2, ... \ ,$$

$$S_n = X_{n1} + ... + X_{nn}, \quad \sum_{j=1}^{n} \sigma_{nj}^2 = 1.$$

Set

$$F_n(x) = P\left(S_n < x\right), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du,$$

$$\Delta_n = \sup_x \left|F_n(x) - \Phi(x)\right|.$$

It is well-known that the following condition (Feller's characteristic)

$$\max_{1 \le j \le n} \sigma_{nj} \to 0, \quad n \to \infty \qquad (U)$$

Is called uniform infinite smallness condition of the sequence of independent r.v.'s. $\{X_{nj},\, j \ge 1\}$. We say that this sequence satisfies Lindeberg condition if for any $\varepsilon > 0$

$$L_n(\varepsilon) = \sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \to 0, \quad n \to \infty. \qquad (L)$$

Here $I(A)$ denotes an indicator of the event A.

It is well-known that under condition (L)

$$\Delta_n \to 0, \quad n \to \infty,$$

what means a central limit theorem (CLT). Lindeberg-Feller's theorem improves above theorem and can be represented as following implication

$$(\mathrm{U})\&(\mathrm{CLT}) \Leftrightarrow (\mathrm{L}),$$

i.e. under condition (U) Lindeberg condition is a necessary condition for CLT.

1 Estimation of numerical characteristics used in CLT

Following V.M.Zolotarev [1] we will call non-classical the limit theorems in which we do not use the condition (U). The first non-classical variants of CLT were proved by Zolotarev in 1967 and Rotar in 1975 (see [1],[2]).

In papers [3], [4] the following estimates of $L_n(\varepsilon)$ $(\varepsilon > 0)$, were obtained.

Theorem A. There exists an absolute constant $C > 0$, such that for any $\varepsilon > 0$

$$\left(1 - e^{-\varepsilon^2/4}\right) \sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \le C \left(\Delta_n + \sum_{j=1}^{n} \sigma_{nj}^4\right). \tag{1}$$

Note. It is obvious that under condition (U) and $\sum\limits_{j=1}^{n} \sigma_{nj}^2 = 1$

$$\sum_{j=1}^{n} \sigma_{nj}^4 \le \max_j \sigma_{nj}^2 \to 0, \quad n \to \infty.$$

Thus (1) implies that if the sequence of independent r.v.'s $\{X_{nj},\, j \ge 1\}$ satisfies CLT (i.e. $\Delta_n \to 0$, $n \to \infty$), then the Lindeberg condition

$$\sum_{j=1}^{n} E(X_{nj}^2 I(|X_{nj}| > \varepsilon)) \to 0$$

holds for any $\varepsilon > 0$ by $n \to \infty$.

Set

$$F_{nj}(x) = P(X_{nj} < x),$$

$\Phi_{nj}(x)$ - distribution function of normal r.v. with parameters $(0,\, \sigma_{nj}^2)$ $(j = 1, 2, ...)$ and for any $\varepsilon > 0$

$$R_n(\varepsilon) = \sum_{j=1}^{\infty} \int\limits_{|x|>\varepsilon} |x||F_{nj}(x) - \Phi_{nj}(x)|dx.$$

Theorem B (V.I.Rotar [2]). The following condition is sufficient and necessary for CLT

$$R_n(\varepsilon) \to 0, \quad n \to \infty \tag{2}$$

For any $\varepsilon > 0$.

Above Theorem B is a nonclassical version of CLT and it generalizes Lindeberg-Feller's theorem. Indeed in Theorem B we do not use the condition (U). The proof the necessity of the condition (2) is based on the following statement (note that a proof of the necessity of the condition (2), given in [2] is rather complicated and it uses the properties of probabilistic metrics).

Theorem 1 *For some* $C = C(\varepsilon)$

$$R_n(\varepsilon) \le C \left(\Delta_n + \sum_{j=1}^{n} \sigma_{nj}^{2s} \right) \tag{3}$$

for all $s \ge 2$.

Proof 2 *For any distribution function* $F(x)$*set*

$$\tilde{F}(x) = \begin{cases} 1 - F(x), & \text{if } x \ge 0; \\ F(x), & \text{if } x < 0. \end{cases}$$

Then for any $k \ge 1$ *we have*

$$\int\limits_{|x|>\varepsilon} |x|^k dF(x) = \int\limits_{\varepsilon}^{\infty} x^k d(-\tilde{F}(x)) + \int\limits_{-\infty}^{-\varepsilon} (-x)^k d(\tilde{F}(x)).$$

The latter can be proved by partial integration.

Using above equation one can prove the following

$$\int\limits_{|x|>\varepsilon} |x|^k dF(x) \ge k \int\limits_{|x|>\varepsilon} |x|^{k-1} \tilde{F}(x) dx, \quad k \ge 1. \tag{4}$$

By definition we have

$$\Phi_{nj}(x) = \Phi\left(x/\sigma_{nj}\right), \tag{5}$$

$$|F(x) - \Phi(x)| \le |\tilde{F}(x) - \tilde{\Phi}(x)| \le \tilde{F}(x) + \tilde{\Phi}(x). \tag{6}$$

Applying (4) and (6) several times one can have the following estimates:

$$R_n(\varepsilon) \le \sum_{j=1}^{n} \left[\int\limits_{|x|>\varepsilon} |x||\tilde{F}_{nj}(x)|dx + \int\limits_{|x|>\varepsilon} |x||\tilde{\Phi}_{nj}(x)|dx \right] \le$$

$$\le \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 dF_{nj}(x) + \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 d\Phi_{nj}(x) =$$

$$= \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 dF_{nj}(x) + \sum_{j=1}^{n} \int\limits_{|x|>\varepsilon} x^2 d\Phi(x/\sigma_{nj}) =$$

$$= L_n(\varepsilon) + \sum_{j=1}^{n} \sigma_{nj}^2 \int\limits_{|x|>\varepsilon/\sigma_{nj}} x^2 d\Phi(x).$$

Now the proof of the theorem 1 follows from Theorem A.

Since $\sum_{j=1}^{n} \sigma_{nj}^2 = 1$, from (3) we have the following implication

$$(U)\&(CLT) \Rightarrow \{R_n(\varepsilon) \to 0,\ n \to \infty\}, \quad \varepsilon > 0.$$

Consequently Theorem B is a generalization of Lindeberg-Feller's theorem.

Now following papers [5], [6] introduce Ibragimov-Osipov-Esseen's characteristic

$$d_n = \sum_{j=1}^{n} \int\limits_{|x|>1} x^2 dF_{nj}(x) + \sum_{j=1}^{n} \left| \int\limits_{|x|\leq 1} x^3 dF_{nj}(x) \right| + \sum_{j=1}^{n} \int\limits_{|x|\leq 1} x^4 dF_{nj}(x) =$$

$$= L_n(1) + L_n^{(2)} + L_n^{(3)}.$$

It is to check that convergence to zero of one of the following sequences $L_n(1)$, $L_n^{(2)}$, $L_n^{(3)}$ does not imply the convergence to zero of other two sequences, for instance the relation $\{L_n(1) \to 0, n \to \infty\}$ does not imply $\{L_n^{(2)} \to 0\}$ or $\{L_n^{(3)} \to 0\}$.

Lemma 1 *The following takes place*

$$\{L_n(\varepsilon) \to 0\} \;\Leftrightarrow\; \{d_n \to 0\}, \quad n \to \infty, \quad \varepsilon > 0.$$

Proof 3 *Let $L_n(\varepsilon) \to 0$, $n \to \infty$ for any $\varepsilon > 0$. In order to prove $d_n \to 0$ it is enough to prove that*

$$\lim_{n\to\infty} \sum_{j=1}^{n} E|X_{nj}|^3 I\left(|X_{nj}| \leq 1\right) = 0.$$

Without loosing generality one can assume that $0 < \varepsilon < 1$. Taking into account the latter

$$I(|X| \leq 1) = I(|X| \leq \varepsilon) + I(\varepsilon < |X| \leq 1).$$

Then

$$L_n^{(2)} \leq \sum_{j=1}^{n} E|X_{nj}|^3 I(|X_{nj}| \leq 1) \leq \varepsilon \sum_{j=1}^{n} EX_{nj}^2 +$$

$$+ \sum_{j=1}^{n} EX_{nj}^2 I(|X_{nj}| \geq \varepsilon) = \varepsilon + L_n(\varepsilon) = \varepsilon + o(1).$$

Analogously we obtain

$$L_n^{(3)} \leq \varepsilon^2 \sum_{j=1}^{n} \sigma_{nj}^2 + \sum_{j=1}^{n} EX_{nj}^2 I\left(\varepsilon < |X_{nj}| \leq 1\right) \leq$$

$$\leq \varepsilon^2 + L_n(\varepsilon) = \varepsilon^2 + o(1), \quad n \to \infty$$

for any $0 < \varepsilon < 1$.

From the last relations we get

$$\{L_n(\varepsilon) \to 0\} \;\Rightarrow\; \{d_n \to 0\}, \quad n \to \infty, \quad \forall \varepsilon > 0.$$

Now let $d_n \to 0$, $n \to \infty$. This means that

$$\lim_{n\to\infty} L_n(1) = \lim_{n\to\infty} L_n^{(2)} = \lim_{n\to\infty} L_n^{(3)} = 0.$$

If $\varepsilon \geq 1$, then

$$L_n(\varepsilon) = \sum_{j=1}^{n} EX_{nj}^2 I(|X_{nj}| > \varepsilon) \leq$$

$$\leq \sum_{j=1}^{n} EX_{nj}^2 I(|X_{nj}| > 1) = L_n(1) \to 0, \quad n \to \infty.$$

If $0<\varepsilon<1$, then

$$L_n(\varepsilon)=\sum_{j=1}^{n} EX_{nj}^2 I(|X_{nj}|>\varepsilon)\le$$

$$\le \varepsilon^{-2}\sum_{j=1}^{n} EX_{nj}^4 I(|X_{nj}|\le 1)+\sum_{j=1}^{n} EX_{nj}^2 I(|X_{nj}|>1)=$$

$$=\varepsilon^{-2}L_n^{(3)}+L_n(1)\to 0,\quad n\to\infty.$$

Consequently taking into account the last relations we have

$$\{d_n\to 0\}\ \Rightarrow\ \{L_n(\varepsilon)\to 0\},\quad n\to\infty,\quad \forall\varepsilon>0.$$

Lemma is proved.

Using the proved lemma it is easy to see that the following version of classic Lindeberg-Feller's theorem takes place.

Theorem 4 *Assume that the condition (U) holds.Then an array $\{X_{nj},\ j\ge 1\}$ satisfies the CLT if and only if*

$$d_n\to 0,\quad n\to\infty. \tag{D}$$

Note that in practice it is easier to check the condition (D) than the Lindeberg condition (L).

We will illustrate an asymptotic behavior of d_n in particular case.Let

$$X_1,\ X_2,\ \ldots,\ X_n,\ldots$$

be a sequence of independent and identically-distributed r.v.'s with common distribution function $F(x)=P(X_1<x)$,

$$\overline{S}_n=X_1+\cdots+X_n.$$

Set $EX_1=0$, $\sigma^2=EX_1^2<\infty$. CLT for this sequence means that

$$P\left(\frac{\overline{S}_n}{\sigma\sqrt{n}}<x\right)\to\Phi(x),\quad n\to\infty,\quad \forall x\in R.$$

In this case we have

$$X_{nj}=\frac{X_j}{\sigma\sqrt{n}},\quad EX_{nj}=0,\quad \sigma_{nj}^2=EX_{nj}^2=\frac{DX_1}{\sigma^2 n}=\frac{1}{n},\quad j=1,2,\ldots.$$

Consequently

$$d_n=\int\limits_{|x|>\sigma\sqrt{n}} x^2dF(x)+\frac{1}{\sigma^3\sqrt{n}}\left|\int\limits_{|x|\le\sigma\sqrt{n}} x^3dF(x)\right|+\frac{1}{\sigma^4\sqrt{n}}\int\limits_{|x|\le\sigma\sqrt{n}} x^4dF(x).$$

Further counting that $0<\varepsilon<1$, we have

$$\frac{1}{\sigma^3\sqrt{n}}\left|\int\limits_{|x|\le\sigma\sqrt{n}} x^3dF(x)\right|\le\frac{1}{\sigma^3\sqrt{n}}\int\limits_{|x|\le\sigma\sqrt{n}} |x|^3dF(x)\le$$

$$\leq \frac{1}{\sigma^3\sqrt{n}} \int\limits_{|x|\leq\varepsilon\sigma\sqrt{n}} |x|^3 dF(x) + \frac{1}{\sigma^3\sqrt{n}} \int\limits_{\varepsilon\leq\left|\frac{x}{\sigma\sqrt{n}}\right|\leq 1} |x|^3 dF(x) \leq$$

$$\varepsilon + \frac{1}{\sigma^2} \int\limits_{|x|\geq\varepsilon\sigma\sqrt{n}} x^2 dF(x) = \varepsilon + o(1).$$

Analogously we obtain

$$\frac{1}{\sigma^4 n} \int\limits_{|x|\leq\sigma\sqrt{n}} x^4 dF(x) \leq$$

$$\leq \frac{1}{\sigma^4 n} \left(\int\limits_{|x|\leq\varepsilon\sigma\sqrt{n}} x^4 dF(x) + + \int\limits_{\varepsilon\leq\left|\frac{x}{\sigma\sqrt{n}}\right|\leq 1} x^4 dF(x) \right) \leq$$

$$\leq \varepsilon^2 + \frac{1}{\sigma^2} \int\limits_{|x|\geq\varepsilon\sigma\sqrt{n}} x^2 dF(x) = \varepsilon^2 + o(1).$$

From above relations one can conclude that $\sigma^2 = DX_1 < \infty$, implies $d_n \to 0$, $n \to \infty$. The latter and theorem 1 imply the following statement (Levi's theorem):

If $\{X_j, j \geq 1\}$ is a sequence of iid r.v.'s with the variance $\sigma^2 = DX_1 < \infty$, then this sequence satisfies CLT.

2 Approximation of the sequence of composition of probabilistic distributions and CLT

CLT can be considered as a particular case of the problem of approximating of the composition s of sequence of probabilistic distributions. Recall that the composition of two probabilistic distributions $F(x)$ and $G(x)$ is defined as following

$$(F * G)(x) = F * G = \int\limits_{-\infty}^{\infty} F(x-u) dG(u) = G * F.$$

Consider two sequences of compositions of probabilistic distributions

$$F_n = F_{n1} * \cdots * F_{nn} = \prod_{j=1}^{n} * F_{nj},$$

$$G_n = G_{n1} * \cdots * G_{nn} = \prod_{j=1}^{n} * G_{nj}.$$

Distribution functions $F_{nj}(x) = F_{nj}$ $(j = 1, 2, \ldots)$ are called components of the composition F_n.

Definition 5 *We say that the sequence of compositions F_n is weakly approximated by the sequence of compositions G_n, if as $n \to \infty$*

$$\int\limits_{-\infty}^{\infty} p(x) d(F_n(x) - G_n(x) dx \to 0$$

For any bounded and continuous function $p(\cdot)$ on R.

Weak approximation of $\{F_n,\ n \geq 1\}$ and $\{G_n,\ n \geq 1\}$ is denoted by

$$F_n - G_n \Rightarrow 0, \quad n \to \infty. \tag{7}$$

Let

$$f_n(t) = \int_{-\infty}^{\infty} e^{itx} dF_n(x), \quad g_n(t) = \int_{-\infty}^{\infty} e^{itx} dG_n(x)$$

Be characteristic functions corresponding to the compositions of the distributions of F_n and G_n.

From the general theory of weak convergence follows that the relation (7) is equivalent to

$$\sup_{|t| \leq T} |f_n(t) - g_n(t)| \Rightarrow 0, \quad n \to \infty \tag{8}$$

for any $T > 0$.

The limit relation (7) generalizes CLT. Indeed in the last case the composition F_n is a distribution function of the sum $S_n = \sum_{j=1}^{n} X_{nj}$, i.e.

$$F_n(x) = P(S_n < x) = \prod_{j=1}^{n} * F_{nj}(x).$$

Set

$$G_{nj}(x) = \Phi\left(x/\sigma_{nj}\right), \quad j = 1, 2, \ldots, n$$

i.e. G_{nj} is a distribution function of normal r.v. with parameters $(0, \sigma_{nj}^2)$. Since $\sum_{j=1}^{n} \sigma_{nj}^2 = 1$, using the property of normal distribution

$$G_n(x) = \prod_{j=1}^{n} * G_{nj} = \prod_{j=1}^{n} * \Phi\left(x/\sigma_{nj}\right) = \Phi(x).$$

Consequently under this choice of distributions $G_n j$ the limit relation (7) considers with CLT.

In the case of arbitrary compositions F_n and G_n the Rotar's characteristic will have the following form:

$$\overline{R}_n(\varepsilon) = \sum_{j=1}^{n} \int_{|x| > \varepsilon} |x|\ |F_{nj}(x) - G_{nj}(x)| dx, \quad \varepsilon > 0.$$

In the book [2] the following theorem is proven.

Theorem C. For the following limit relation

$$F_n - G_n \Rightarrow 0, \quad n \to \infty.$$

It is enough:

$$\overline{R}_n(\varepsilon) \Rightarrow 0, \quad n \to \infty$$

for any $\varepsilon > 0$.

The Lindeberg-Feller's theorem and Theorem B were generalized in the papers by the author [7] using the "closeness of distributions" characteristic

$$\alpha_n(T) = \sum_{j=1}^{n} \sup_{|t| \leq T} \left| f_{nj}(t) - g_{nj}(t) \right|,$$

where $f_{nj}(t)$ and $g_{nj}(t)$ are characteristic functions corresponding to the distributions F_{nj} and G_{nj} $(j = 1, 2, \ldots)$.

Theorem 6 *Let*

$$\int_{-\infty}^{\infty} x dF_{nj}(x) = \int_{-\infty}^{\infty} x dG_{nj}(x) = 0,$$

$$\int_{-\infty}^{\infty} x^2 dF_{nj}(x) = \int_{-\infty}^{\infty} x^2 dG_{nj}(x) = 0, \quad j = 1, 2, \ldots.$$

If the following condition holds

$$\alpha_n(T) \to 0, \quad n \to \infty, \quad \forall T > 0, \tag{9}$$

then $F_n - G_n \Rightarrow 0, \quad n \to \infty$.

Proof 7 *The proof of this theorem is almost obvious because for all complex numbers we have*

$$|a_k| \le 1, \quad |b_k| \le 1, \quad k = 1, 2, \ldots$$

$$\left| \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right| \le \sum_{k=1}^{n} |a_k - b_k|. \tag{10}$$

From (10)

$$\left| \prod_{j=1}^{n} f_{nj}(t) - \prod_{j=1}^{n} g_{nj}(t) \right| \le \sum_{j=1}^{n} \left| f_{nj}(t) - g_{nj}(t) \right|.$$

Under condition (9) the last inequality implies the limit relation (8). heorem6 is proven.

3 Non-classical versions of CLT based on the Ibragimov-Osipov-Esseen characteristic

Introduce the following notation:

$$R_n^{(1)} = R_n(1) = \sum_{j=1}^{n} \int_{|x| \ge 1} |x| \ \left| F_{nj}(x) - \Phi_{nj}(x) \right| dx,$$

$$R_n^{(2)} = \sum_{j=1}^{n} \int_{|x| \le 1} x^2 \left| F_{nj}(x) - \Phi_{nj}(x) \right| dx,$$

$$R_n^{(3)} = \sum_{j=1}^{n} \int_{|x| \le 1} |x|^3 \left| F_{nj}(x) - \Phi_{nj}(x) \right| dx,$$

$$\delta_n = R_n^{(1)} + R_n^{(2)} + R_n^{(3)}.$$

It is natural to call δ_n as a "difference" characteristic of Ibragimov-Osipov-Esseen and it takes into account the "closeness" of distribution functions F_{nj} to normal distribution function with parameters $(0, \sigma_{nj}^2)$. It is worth to note that the existence of d_n implies the existence of δ_n and this is based on the following relations

$$|F(x) - \Phi(x)| \le |\tilde{F}(x) - \tilde{\Phi}(x)| \le \tilde{F}(x) + \tilde{\Phi}(x).$$

Recall that $\tilde{F}(x) = 1 - F(x)$ for $x > 0$, $\tilde{F}(x) = F(x)$ for $x \leq 0$. In part,

$$\int\limits_{-\infty}^{\infty} |x|\ |F(x) - \Phi(x)|dx \leq \int\limits_{-\infty}^{\infty} x\tilde{F}(x)dx + \int\limits_{-\infty}^{\infty} x\tilde{\Phi}(x)dx = \\ = 2\left(\int\limits_{-\infty}^{\infty} x^2 dF(x) + \int\limits_{-\infty}^{\infty} x^2 d\Phi(x)\right). \tag{11}$$

Theorem 8 *The following relations take place*

$$\{\delta_n \to 0\} \ \Leftrightarrow \ \{R_n(\varepsilon) \to 0, \ \ \varepsilon > 0\}, \quad n \to \infty.$$

Proof 9 *Let for any* $\varepsilon > 0$,

$$R_n(\varepsilon) \to 0, \quad n \to \infty.$$

holds.Then

$$R_n^{(1)} = R_n(1) \to 0, \quad n \to 0. \tag{12}$$

Moreover for $0 < \varepsilon < 1$ *from (11) we have*

$$R_n^{(2)} = \sum_{j=1}^{n} \int\limits_{|x| \leq \varepsilon} x^2 |F_{nj}(x) - \Phi_{nj}(x)|dx + \sum_{j=1}^{n} \int\limits_{\varepsilon \leq |x| \leq 1} x^2 |F_{nj}(x) - \Phi_{nj}(x)|dx \leq \\ \leq \varepsilon \sum_{j=1}^{n} \int\limits_{-\infty}^{\infty} |x||F_{nj}(x) - \Phi_{nj}(x)|dx + \sum_{j=1}^{n} \int\limits_{|x| > \varepsilon} |x||F_{nj}(x) - \Phi_{nj}(x)|dx \leq \\ \leq 2\varepsilon \sum_{j=1}^{n} \sigma_{nj}^2 + R_n(\varepsilon) \to 2\varepsilon + o(1), \tag{13}$$

$$R_n^{(3)} = \sum_{j=1}^{n} \int\limits_{|x| \leq 1} |x|^3 |F_{nj}(x) - \Phi_{nj}(x)|dx = \\ = \sum_{j=1}^{n} \int\limits_{|x| \leq \varepsilon} |x|^3 |F_{nj}(x) - \Phi_{nj}(x)|dx + \sum_{j=1}^{n} \int\limits_{\varepsilon \leq |x| \leq 1} |x|^3 |F_{nj}(x) - \Phi_{nj}(x)|dx \leq \\ \leq \varepsilon^2 \sum_{j=1}^{n} \int\limits_{-\infty}^{\infty} |x||F_{nj}(x) - \Phi_{nj}(x)|dx + \sum_{j=1}^{n} \int\limits_{|x| > \varepsilon} |x||F_{nj}(x) - \Phi_{nj}(x)|dx \leq \\ \leq 2\varepsilon^2 \sum_{j=1}^{n} \sigma_{nj}^2 + R_n(\varepsilon) \to 2\varepsilon^2 + o(1). \tag{14}$$

From relations (12), (13) and (14) we find that

$$d_n \to 0, \quad n \to \infty. \tag{15}$$

Now assume that relation (15) holds. Then for $0 < \varepsilon < 1$

$$R_n(\varepsilon) = \sum_{j=1}^{n} \int\limits_{\varepsilon < |x| \leq 1} |x||F_{nj}(x) - \Phi_{nj}(x)|dx + \sum_{j=1}^{n} \int\limits_{|x| > 1} |x||F_{nj}(x) - \Phi_{nj}(x)|dx \leq$$

$$\leq \varepsilon^{-1}\sum_{j=1}^{n}\int\limits_{|x|\leq 1} x^2|F_{nj}(x)-\Phi_{nj}(x)|dx + R_n(1) =$$

$$= \varepsilon^{-1}R_n^{(2)} + R_n^{(1)} \to 0, \quad n\to\infty. \tag{16}$$

Further for $\varepsilon > 1$ we obviously have

$$R_n(\varepsilon) \leq R_n(1) = R_n^{(n)} \to 0, \quad n\to\infty. \tag{17}$$

Thus from the relations (16), (17) it follows that

$$R_n(\varepsilon), \quad \varepsilon > 0, \quad n\to\infty.$$

Theorem 8 is proven.

Now we will prove the following statement.

Theorem 10 *CLT takes place if and only if*

$$\delta_n \to 0, \quad n\to\infty. \tag{18}$$

Necessity of the condition (18) follows from Theorem B taking into account theorem 8.

We will give a proof of the sufficiently of the condition (18). We will use the theorem 6. In this theorem we set

$$\Phi_{nj}(x) = G_{nj}(x) = \Phi\left(x/\sigma_{nj}\right), \quad j = 1,2,\ldots.$$

Then the corresponding characteristic function has the form

$$g_{nj}(t) = e^{-\sigma_{nj}^2 t^2/2}, \quad j=1,2,\ldots.$$

Estimate

$$\alpha_n(T) = \sum_{j=1}^{n}\sup_{|t|\leq T}\left|f_{nj}(t) - e^{-\sigma_{nj}^2 t^2/2}\right|, \quad T>0.$$

For any $j = 1,2,\ldots$ we have

$$f_{nj}(t) - e^{-\sigma_{nj}^2 t^2/2} =$$

$$= \int\limits_{-\infty}^{\infty}\left(e^{itx} - 1 - itx - \frac{itx^2}{2}\right) d\left(F_{nj}(x) - \Phi_{nj}(x)\right). \tag{19}$$

Above we used equations $(j = 1,2,\ldots)$

$$\int\limits_{-\infty}^{\infty} x dF_{nj}(x) = \int\limits_{-\infty}^{\infty} x d\Phi_{nj}(x) = 0,$$

$$\int\limits_{-\infty}^{\infty} x^2 dF_{nj}(x) = \int\limits_{-\infty}^{\infty} x^2 d\Phi_{nj}(x) = \sigma_{nj}^2.$$

In the integral (19) using integration by part we obtain

$$\left|f_{nj}(t) - e^{-\sigma_{nj}^2 t^2/2}\right| = |t|\left|\int\limits_{-\infty}^{\infty}(e^{itx} - 1 - itx)(F_{nj}(x) - \Phi_{nj}(x))dx\right| \leq$$

$$\leq |t| \left(I_{1j}(t) - I_{2j}(t) \right), \tag{20}$$

where

$$I_{1j}(t) = \left| \int\limits_{|x|\leq 1} \left(e^{itx} - 1 - itx - \frac{(itx)^2}{2} \right) (F_{nj}(x) - \Phi_{nj}(x)) dx \right| +$$

$$+ \left| \int\limits_{|x|\leq 1} \left(\frac{(itx)^2}{2} \right) (F_{nj}(x) - \Phi_{nj}(x)) dx \right|,$$

$$I_{2j}(t) = \left| \int\limits_{|x|>1} \left(e^{itx} - 1 - itx \right) (F_{nj}(x) - \Phi_{nj}(x)) dx \right|.$$

It is easy to see the that the following estimates hold:

$$I_{1j}(t) \leq \frac{|t|^3}{3!} \int\limits_{|x|\leq 1} |x|^3 |F_{nj}(x) - \Phi_{nj}(x)| dx$$

$$+ \frac{t^2}{2} \int\limits_{|x|\leq 1} x^2 |F_{nj}(x) - \Phi_{nj}(x)| dx, \tag{21}$$

$$I_{2j}(t) \leq 2|t| \int\limits_{|x|>1} |x| |F_{nj}(x) - \Phi_{nj}(x)| dx \tag{22}$$

Now from the relations (20), (21) and (22) we have

$$\alpha_n(T) \leq 2 \max(t^2, t^4) \left(\sum_{j=1}^{n} \int\limits_{|x|\leq 1} |x|^3 |F_{nj} - \Phi_{nj}| dx + \right.$$

$$\left. + \sum_{j=1}^{n} \int\limits_{|x|\leq 1} |x|^2 |F_{nj} - \Phi_{nj}| dx + \sum_{j=1}^{n} \int\limits_{|x|>1} |x| |F_{nj} - \Phi_{nj}| dx \right) \leq$$

$$\leq 2 \max(t^2, t^4) \cdot \delta_n. \tag{23}$$

The proof of the theorem 10 follows from the relation (23) and the theorem 8. From the last estimate it follows that the condition

$$\alpha_n(T) \to 0, \quad T > 0, \quad n \to \infty$$

in theorem 6 in the case of CLT is a necessary condition.

From the proof of the theorem 8 and 10 one can prove the following statements.

Theorem 11 *For some $C > 0$*

$$R_n(\varepsilon) \leq C(\delta_n + \max_j \sigma_{nj}^2)$$

for any $\varepsilon > 0$.

Theorem 12 *For some $C > 0$*

$$\delta_n \leq C(\Delta_n + \max_j \sigma_{nj}^2).$$

Theorem 13 *For some $C > 0$*

$$R_n(\varepsilon) \le C(\Delta_n + \max_j \sigma_{nj}^2)$$

for any $\varepsilon > 0$.

Theorem 14 *For any $T > 0$ and some $C > 0$*

$$\alpha_n(T) \le C(\Delta_n + \max_j \sigma_{nj}^2).$$

Conclusion

Theorems 11-14 generalize given above Lindeberg-Feller's theorem and are analogous of Theorem A in terms of different numerical characteristics used in the proofs of the non-classical versions of CLT. Theorem 10 is a generalization of Theorem B (V.I.Rotar) , because the condition (2) implies the following limit relation

$$\delta_n \to 0, \quad n \to \infty.$$

It is worth to note that the last condition can be checked easier than the following condition

$$R_n(\varepsilon) \to 0, \quad n \to \infty, \quad \forall \varepsilon > 0.$$

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