Recursive estimation of one-dimensional parameter of Compound Poisson process*

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Abstract

Recursive estimation procedure of a one-dimensional parameter of Levy measure of Compound Poisson process is introduced and their asymptotic properties are investigated.

The object of our investigation is a parameter filtered statistical model

$$\mathcal{E} = \big(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, (P_\theta, \theta \in R)\big) \tag{1}$$

associated with one-dimensional $\mathbb{F}$-adapted RCLL process $X = (X_t)_{t\geq 0}$ in the following way: for each $\theta \in R$, P_θ is assumed to be the unique measure on $(\Omega, \mathcal{F})$ such that under P_θ, $X = (X_t)_{t\geq 0}$ is a semimartingale the triplet of predictable characteristics $(B_\theta, 0, \nu_\theta)$, where $B_\theta(t) = \lambda_t \int_R x I_{(|x|\leq 1)} \nu(\theta, dx)$, $\nu_\theta(dt, dx) = \lambda\, dt\, \nu(\theta; dx)$, where $\lambda > 0$, $\nu(\theta, \cdot)$ is probability measure with $\int_R x^2 \nu(\theta, dx) < \infty$.

It is obvious that under P_θ, $X = (X_t)_{t\geq 0}$ is a Compound Poisson process that can be written in the following form:

$$X_t = \sum_{i=1}^{N_t} \xi_i,$$

where $N = (N_t)_{t\geq 0}$ is a Standard Poisson process with intensity $\lambda > 0$, and $\xi = (\xi_n)_{n\geq 1}$ is the sequence of i.i.d. random variables with the probability distribution $\nu(\theta, \cdot)$ (see, e.g., [1]).

Our aim is to construct recursive estimation procedure for unknown parameter $\theta \in R$.

Suppose that for each pair (θ, θ') the measures $\nu(\theta, \cdot)$ and $\nu(\theta', \cdot)$ are equivalent. Fix some $\widetilde{\theta} \in R$ and denote $P_{\widetilde{\theta}} := P$, $\nu_{\widetilde{\theta}} := \nu$, $\nu(\widetilde{\theta}; x) = \nu(\cdot)$. Then

$$P_\theta \overset{loc}{\sim} P, \quad \theta \in R,$$

and the local density process $\rho_t(\theta) = \frac{dP_{\theta,t}}{dP_t}$ can be represented as the Dolean esponential

$$\rho_t(\theta) = \mathcal{E}_t(M(\theta)),$$

where $M(\theta) = (Y(\theta) - 1) * (\mu - \nu)$, where $Y(\theta, x) = \frac{d\nu(\theta,x)}{d\nu(x)}$.

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Further, assume that the density $Y(\theta, x)$ is continuously differentiable in θ for each $x \in R$, and differentiability under integral signs is possible. Note that under assumptions listed above the model (1) is regular in the sense given in [2].

It is not hard to observe that

$$L_t(\theta) := \frac{\partial}{\partial\theta} \ln \rho_t(\theta) = \frac{\dot{Y}(\theta)}{Y(\theta)} * (\mu - \nu_\theta) = \Phi(\theta) * (\mu - \nu_\theta) \quad \Big(\Phi(\theta) = \frac{\dot{Y}(\theta)}{Y(\theta)}\Big). \tag{2}$$

Hence, the maximum likelihood equation is

$$L_t(\theta) = \Phi(\theta) * (\mu - \nu_\theta)_t = 0. \tag{3}$$

Remark 1. The problem of solvability of Eq. (3) in more general setting is studied in [3].

Eq. (3) can be rewritten in the equivalent form

$$\sum_{n=1}^{N_t} \frac{\dot{Y}(\theta, \xi_n)}{Y(\theta, \xi_n)} = 0 \quad \Longrightarrow \quad \widehat{\theta}_t - \text{MLE}. \tag{3$'$}$$

So, for each $t > 0$, we need to solve Eq. (3) or (3$'$) which is not easy task (in general).

Instead in [2] we proposed the recursive procedure to obtain the process $\theta = (\theta_t)_{t\geq 0}$ (recursive estimate) with the same asymptotic properties as MLE $\widehat{\theta}_t$, as $t \to \infty$, P_θ-a.s.

To develop this procedure first of all assume that

$$I(\theta) := \int_R \Phi^2(x, \theta)\nu(\theta, dx) < \infty.$$

Then $L(\theta) = (L_t(\theta), t \geq 0) \in M^2_{loc}(P_\theta)$ and the Fisher information process is

$$I_t(\theta) = \langle L(\theta), L(\theta)\rangle_t = \lambda t I(\theta).$$

Denote $\gamma_t(\theta) = I_t^{-1}(\theta)$. The recursive estimation procedure, SDE (3.4) of [4] in the case under consideration is of the following form:

$$\begin{aligned} \theta_t = \theta_0 &+ \int_0^t \int_R \gamma_s(\theta_{s-})\Phi(\theta_{s-}, x)\Big(1 - \frac{Y(\theta_{s-}, x)}{Y(\theta, x)}\Big)\nu(ds, dx) \\ &+ \int_0^t \int_R \gamma_s(\theta_{s-})\Phi(\theta_{s-}, x)(\mu - \nu_\theta)(ds, dx). \end{aligned} \tag{4}$$

Remark 2. Although Eq. (4) is equivalent to the following equation

$$\theta_t = \theta_0 + \sum_{n=1}^{N_t} \frac{\Phi(\theta_{n-1}, \xi_n)}{nI(\theta)}\,, \tag{4$'$}$$

we prefer the form of Eq. (4) to investigate asymptotic properties of θ_t, as $t \to \infty$, P_θ-a.s., based on results concerning asymptotic behaviour of solutions of Robbins–Monro (RM) type SDE.

The Robbins–Monro type SDE

$$z_t = z_0 + \int_0^t H_s(z_{s-})dK_s + \int_0^t M(ds, z_{s-}), \tag{5}$$

where $H_s(0) = 0$, $H_s(u)u < 0$, $u \neq 0$, was introduced in [5]. In [5], [6], the asymptotic behaviour of $z = (z_t)_{t\geq 0}$, as $t \to \infty$, P_θ-a.s. was investigated.

Assume that for each $x \in R$ the function $Y(\theta, x)$ is strongly monotone in θ. Denote $z_t = \theta_t - \theta$, then Eq. (4) becomes

$$\begin{aligned} z_t = z_0 &+ \int_0^t \int_R \gamma_s(\theta + z_{s-})\Phi(\theta + z_{s-}, x)\left(1 - \frac{Y(\theta + z_{s-}, x)}{Y(\theta, x)}\right)\nu_\theta(ds, dx) \\ &+ \int_0^t \int_R \gamma_s(\theta + z_{s-})\Phi(\theta + z_{s-}, x)(\mu - \nu_\theta)(ds, dx) \end{aligned} \tag{5'}$$

and is of the form (5) with $K_t = \lambda t$,

$$H_t(u) = \int_R \gamma_t(\theta + u)\Phi(\theta + u, x)\left(1 - \frac{Y(\theta + u, x)}{Y(\theta, x)}\right)\nu(\theta, dx), \tag{6}$$

$$M(u) = [M_t(u), t \geq 0] = \left[\int_0^t \int_R \gamma_s(\theta + u)\Phi(\theta + u, x)(\mu - \nu_\theta)(ds, dx),\ t \geq 0\right]. \tag{7}$$

Denote

$$h_t(u) = \frac{d\langle M(u), M(u)\rangle_t}{dK_t} = \gamma_t^2(\theta + u)\int_R \Phi^2(\theta + u, x)\nu(\theta, dx) = \gamma_t^2(\theta + u)I(\theta + u).$$

Hence, $(5')$ is the special case of the RM type SDE (5) with objects $H(u) = (H_t(u))_{t\geq 0}$ and $M(u) = (M(t, u))_{t\geq 0}$ specified by Eqs. (6) and (7), respectively.

Therefore one can use the results about asymptotic behaviour of solution $z = (z_t)_{t\geq 0}$ of general SDE (5) to establilsh asymptotic behaviour of solution of SDE $(5')$, as $t \to \infty$.

Namely, one can use Theorem 3.1 of [5] to derive sufficient conditions for the convergence: $z_t \to 0$, as $t \to \infty$, P_θ-a.s., for all $\theta \in R$ (recall that from now $z = (z_t)_{t\geq 0}$ is the solution of SDE $(5')$). Further, sufficient conditions for the convergence: for all δ, $0 < \delta < \frac{1}{2}$, $I_t^\delta z_t \to 0$ (rate of convergence), as $t \to \infty$, P_θ-a.s., can be obtained from Theorem 2.1 of [6] and, finally, to establish the asymptotic distribution of $I_t^{1/2}(\theta)z_t$, as $t \to \infty$ (under measure P_θ), one can use Theorem 3.1 of [6].

As an illustration, in the present work we restrict ourselves by results concerning the convergence: $z_t \to 0$, as $t \to \infty$, P_θ-a.s., rate of convergence, to avoid complex notation needed to state conditions for the validity of asymptotic expansion of $I_t^{1/2} z_t$ (see Eq. (3.1) from [?], with $R_t \xrightarrow{P_\theta} 0$, as $t \to \infty$.

Note that this convergence is equivalent to the strong consistency of recursive estimate $(\theta_t)_{t\geq 0}$ given by (4) or $(4')$, that is $\theta_t \to \theta$, as $t \to \infty$, P_θ-a.s.

Theorem 1. *Let the following conditions be satisfied: for all* $\theta \in R$

(i) $I^{-1}(\theta+u) < c(\theta)(1+u^2)$, $c(\theta) > 0$;

(ii) *for each* ε, $\varepsilon > 0$,

$$\inf_{\varepsilon \le |u| \le \frac{1}{\varepsilon}} \left| u I^{-1}(\theta+u) \int_R \Phi(\theta+u,x)\Big(1 - \frac{Y(\theta+u,x)}{Y(\theta,x)}\Big)\nu(\theta,dx)\right| > 0.$$

Then $z_t \to 0$, *as* $t \to \infty$, P_θ*-a.s.*

Proof. Condition (A) of Theorem 3.1 from [5] follows from the strong monotonicity of $Y(\theta,x)$ w.r.t. θ, for all $x \in R$.

Condition (B) of Theorem 3.1 of [5] is also satisfied, since

$$h_t(u) = \gamma_t^2(\theta+u) I(\theta+u) = \frac{1}{\lambda^2 t^2 I^2(\theta+u)}\, I(\theta+u) = I^{-1}(\theta+u)\,\frac{1}{\lambda^2 t^2}\,.$$

Therefore

$$h_t(u) \le B_t(1+u^2),$$

with $B_t = \frac{c(\theta)}{\lambda^2}\,\frac{1}{t^2}$ and $\int_0^\infty B_s\,ds = \infty$.

Condition (I) of Theorem 3.1 from [5] is satisfied, since

$$\int_0^\infty \inf_{\varepsilon \le |u| \le \frac{1}{\varepsilon}} |u H_t(u)|\,dK_t = \int_0^\infty \inf \left| u\gamma_t(\theta+u) \int_R \Phi(\theta+u,x)\Big(1 - \frac{Y(\theta+u,x)}{Y(\theta)}\Big)\nu(\theta,dx)\right| \lambda\,dt$$

$$= \inf_{\varepsilon \le |u| \le \frac{1}{\varepsilon}} \left| u I^{-1}(\theta+u) \int_R \Phi(\theta+u,x)\Big(1 - \frac{Y(\theta+u,x)}{Y(\theta)}\Big)\nu(\theta,dx)\right| \int_0^\infty \frac{dt}{t} = \infty. \qquad \square$$

Below we assume that $z_t \to 0$, as $t \to \infty$, P_θ-a.s.

Denote

$$\beta_t = -\lim_{u\to 0}\frac{H_t(u)}{u}\,, \quad \beta(u) = \begin{cases} -\frac{H_t(u)}{u}, & u \ne 0, \\ \beta_t, & u = 0. \end{cases}$$

Theorem 2. *Suppose that for each* δ, $0 < \delta < 1$, *the following conditions are satisfied:*

(i) $\displaystyle\int_0^\infty [\delta_t - 2\beta_t(u)]^+ dt < \infty$;

(ii) $\displaystyle\int_0^\infty I_t^\delta(\theta) h_t(z_{t-}, z_{t-})\,dt < \infty$ P_θ*-a.s., where* $h_t(u,v) = \dfrac{d\langle M(u), M(v)\rangle_t}{dK_t}$.

Then $I_t^\delta(\theta) z_t^2 \to 0$, *as* $t \to \infty$, P_θ*-a.s.*

Proof. It is enough to note that in Theorem 2.1 of [6] we must take $\gamma_t = t$ and $r_t^\delta = \frac{\delta}{t}$. $\square$

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